Influence of Atomic Binding on the Recoil Distribution in Electron-Field Pair Production*

CHARLES KAUFMAN
Physics Department, The Pennsylvania State University, University Park, Pennsylvania
(Received 28 February 1964)

The recoil-momentum distribution in high-energy triplet production in hydrogen, taking into account the initial atomic binding of the electron, is calculated. The method of Wheeler and Lamb is used, with the final atomic state specified to be an outgoing electron (Coulomb wave). Explicit recoil distributions for various photon energies from 84 to 2000 mc^2 are obtained, and a binding correction to the total triplet cross section of Borsellino is calculated.

I. INTRODUCTION

CALCULATIONS of triplet production have been of two types: pair production in the field of a free electron[1,2] and pair production in the field of an electron bound in an atom.[3] The relevant experiments, of course, have bound electrons as targets, so the free electron calculations must either be somewhat modified before being compared with experiment,[4,5] or else be considered as approximations, valid only in the region of high momentum transfer to the recoil electron[6] ($\kappa \gg mc/137$), where the electron may be considered completely free.[7] On the other hand, the calculation which does include the effect of binding forces (Wheeler and Lamb's) contains several approximations also. It does not allow for the possibility of electron exchange, and ignores retardation effects. The calculation is a valid approximation, therefore, in the region of low recoil momentum ($\kappa \ll mc$).

The two types of calculations are thus complementary and a simple combination of the two should be a good description of triplet production. That is, the distribution in momentum of the recoil electron should be obtained from the free-electron calculations for high κ, and the Wheeler-Lamb calculation for low κ. The total cross section is then the integral of this composite recoil distribution over all momenta. The recoil distribution for the free-electron case has been calculated,[1,8,9] and we will calculate here the distribution implied by the Wheeler-Lamb calculation.

We have calculated the recoil momentum distribution, for triplet production in hydrogen, at several photon energies. The calculation of the recoil distribution is described in Sec. II, and the results are presented in Sec. III.

II. THE RECOIL DISTRIBUTION

In this section we calculate the cross section for the following process: a photon of momentum $\mathbf{k}$ creates an electron and a positron, of momenta $\mathbf{p}_-$ and $\mathbf{p}_+$, directions (θ_-,φ_-) and (θ_+,φ_+), and energies E_- and E_+, in an external atomic Coulomb field, which absorbs momentum $\mathbf{q}\equiv\mathbf{k}-\mathbf{p}_+-\mathbf{p}_-$ and energy $q_0\equiv k-E_+-E_-$. The atomic state changes from Ψ_i to Ψ_f. We do the calculation in second-order perturbation theory of the interaction Hamiltonian

$$H_I=\int d^3r\left[e(\psi^+\alpha\psi)\cdot\mathbf{A}+e\psi^+\left(\frac{Ze}{r}-\sum_i\frac{e}{|\mathbf{r}-\mathbf{r}_i|}\right)\psi\right],$$

and use exact atomic wave functions Ψ_i and Ψ_f.

When the intermediate state sum is taken, the pair production amplitude is

$$T_{fi}=e^3\left(\frac{2\pi}{k}\right)^{1/2}\frac{4\pi}{q^2}\langle\Psi_f(\mathbf{r}_i)|Z-\sum_i e^{i\mathbf{q}\cdot\mathbf{r}_i}|\Psi_i(\mathbf{r}_i)\rangle$$
$$\times\left\{u_-^+(\mathbf{p}_-)\left[\frac{2\mathbf{e}\cdot\mathbf{p}_--\alpha\cdot\mathbf{e}(k+\mathbf{k}\cdot\alpha)}{2(kE_--\mathbf{k}\cdot\mathbf{p}_-)}+\frac{-2\mathbf{e}\cdot\mathbf{p}_+(k+\mathbf{k}\cdot\alpha)\alpha\cdot\mathbf{e}}{2(kE_+-\mathbf{k}\cdot\mathbf{p}_+)}\right]u_+(-\mathbf{p}_+)\right\},$$

where $u_-(\mathbf{p}_-)$ is a positive energy spinor, $u_+(-\mathbf{p}_+)$ is a negative energy spinor, and $\mathbf{e}$ is the polarization of the incident photon. In the case of "coherent" pair production, the atom is not excited and Ψ_f in the above equation becomes Ψ_i. Except for this difference, the calculation is identical with that of screened nuclear field pair production.[10] The result, for the total cross section, after summing over final spins and averaging

* Supported in part by the U. S. Atomic Energy Commission.
[1] A Borsellino, Nuovo Cimento 4, 112 (1947).
[2] V. Votruba, Bull. Intern. Acad. Tcheque Sci. 49, 19 (1948).
[3] J. A. Wheeler and W. E. Lamb, Phys. Rev. 55, 858 (1939); **101**, 1836 (1956).
[4] D. C. Gates, R. W. Kenney, and W. P. Swanson, Phys. Rev. **125**, 1310 (1962).
[5] D. C. Gates, Ph. D. thesis, University of California, UCRL-9390, 1960 (unpublished).
[6] We distinguish between $\mathbf{q}$, the total momentum transferred to the recoiling system, and κ, that part of $\mathbf{q}$ taken by the recoil electron.
[7] E. L. Hart, G. Cocconi, T. Cocconi, and J. M. Sellen, Phys. Rev. **115**, 678 (1959).
[8] K. S. Suh and H. A. Bethe, Phys. Rev. **115**, 672 (1959).
[9] J. Joseph and F. Rohrlich, Rev. Mod. Phys. **30**, 354 (1958).
[10] H. A. Bethe and W. Heitler, Proc. Roy. Soc. (London) **A146**, 83 (1934).

over initial polarizations, is

$$\sigma(k)=\sum_{\Psi_f}\int dE_+d\Omega_+d\Omega_-|\langle\Psi_f|Z-\sum_i e^{i\mathbf{q}\cdot\mathbf{r}_i}|\Psi_i\rangle|^2 \cdot\frac{\sigma_{\mathrm{BH}}(\theta_+,\varphi_+,\theta_-,\varphi_-)}{Z^2},$$

where $\sigma_{\mathrm{BH}}(\theta_+,\varphi_+,\theta_-,\varphi_-)$ is the Bethe-Heitler differential cross section for nuclear field pair production. To obtain the result in this form, we have assumed $q_0^2\ll\mathbf{q}^2$ which is valid as long as $(v/c)\ll1$ for the recoil electron.

The four integrations over the directions of the pair electrons are done by using the transformation, given by Bethe,[11] to a new set of variables, one of which is q. Three of the integrations can then be done analytically (in the approximation $E_\pm$, $k\gg mc^2$), but the q integral cannot.

We now calculate the cross section for transitions to a particular final atomic state, an ionized atom plus an outgoing electron of momentum $\boldsymbol{\kappa}$. This step is avoided in Wheeler and Lamb's work by using a sum rule to obtain the total cross section. For triplet production in hydrogen

$$\langle\Psi_f|Z-\sum_i e^{i\mathbf{q}\cdot\mathbf{r}_i}|\Psi_i\rangle=\langle\psi_\kappa^-(\mathbf{r})|-e^{i\mathbf{q}\cdot\mathbf{r}}|\psi_0(\mathbf{r})\rangle,$$

where $\psi_0(\mathbf{r})$ is the ground-state wave function, and $\psi_\kappa^-(\mathbf{r})$ is the wave function for an electron of momentum $\boldsymbol{\kappa}$ in a Coulomb field. The sum over all final states becomes an integral over all $\boldsymbol{\kappa}$ (neglecting nonionizing excitations), and the integrations over all directions of $\boldsymbol{\kappa}$ can be done analytically.

After the transformation of variables and the (five) analytic integrations, the total cross section is

$$\begin{aligned}\sigma(k)&=\int d\kappa\sigma(k,\kappa)\\&=\frac{8e^4}{137}\int\kappa^2d\kappa\int\frac{dE_+}{k^3}\int_\delta^{q_{\max}}\frac{2q}{q^3}\\&\quad\times\Bigg\{\frac{\zeta(E_+^2+E_-^2)-E_+E_-}{2[\zeta(1+\zeta)]^{1/2}}\log\left[2\frac{\delta}{q}\zeta(1+\zeta)-\zeta-(\zeta(1+\zeta))^{1/2}\left(1-4\zeta\frac{\delta}{q}+4\zeta(1+\zeta)\frac{\delta^2}{q^2}\right)^{1/2}\right]\\&\quad-\frac{(E_+^2+E_-^2)(q-\delta)^2}{64(1+\zeta)}\frac{\delta}{q}[16(1+\zeta)-\delta q(q+\delta)^2]+E_+E_-\left(\frac{1}{2}+\frac{\frac{1}{2}-(\delta/q)(1+\zeta)}{[1-4\zeta(\delta/q)+4\zeta(1+\zeta)(\delta/q)^2]^{1/2}}\right)\Bigg\}\\&\quad\times\left\{2^8a_0^{-6}\frac{q^2}{\kappa}\exp\left[\frac{2}{\kappa a_0}\tan^{-1}\left(\frac{-2\kappa/a_0}{q^2-\kappa^2+a_0^{-2}}\right)\right]\frac{q^2+[\kappa^2+a_0^{-2}]/3}{(1-e^{-2\pi/\kappa a_0})[(q-\kappa)^2+a_0^{-2}]^3[(q+\kappa)^2+a_0^{-2}]^3}\right\},\end{aligned}$$

where a_0 is the first Bohr radius, $\delta\equiv k-p_+-p_-$, $\zeta\equiv(q^2-\delta^2)/4$, and $q_{\max}$ is of the order of k. The q and E_+ integrals were done numerically, and the resulting function of k and κ is the triplet recoil distribution.

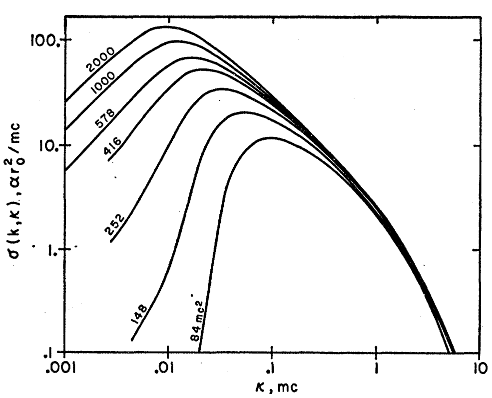

FIG. 1. The recoil momentum distributions for triplet production in hydrogen, according to the Wheeler-Lamb method, for various initial photon energies.

[11] H. A. Bethe, Proc. Cambridge Phil. Soc. 30, 524 (1934).

III. RESULTS

The recoil distributions, in hydrogen, for incident photon energies from 84 to 2000 mc^2, are shown in Fig. 1. In most of the region $\kappa<mc/10$, where the approximations made here are valid, the cross sections are significantly smaller than the free-electron recoil distributions. For κ between 0 and $\delta=2/k$, the "minimum momentum transfer," the cross sections are not zero as they are in the free-electron case. The electron shares the recoil with the rest of the atom, so all values of κ down to $\kappa=0$ are possible.

The calculation was checked by integrating the recoil distributions over all κ. The result agrees with the Wheeler-Lamb total cross section[12] to within the numerical error (about 1%). This agreement also shows that nonionizing excitations of the atom are unimportant, since they are included by the Wheeler-Lamb sum rule, but omitted in the present calculation.

The total triplet cross section is calculated by

[12] The Wheeler-Lamb total cross section was obtained by numerical integration of their differential cross section using the screening functions given in their paper.

TABLE I. Binding corrections to Borsellino's total triplet cross section. The correction c_1 is that calculated here; c_2 is the correction used by Gates. (See Refs. 4 and 5.) k is the photon energy.

k (mc^2)	c_1 (αr_0^2)	c_2 (αr_0^2)
416	$-0.1\ \pm0.1$	0.05
578	-0.01 ± 0.1	0.16
1000	0.42 ± 0.1	0.47
2000	1.42 ± 0.1	1.21

integrating this recoil distribution from $\kappa=0$ up to an intermediate value of the momentum, at which both approximations $\kappa\ll mc$, $\kappa\gg mc/137$ are valid, and integrating the Borsellino recoil distribution above that.[13] Figures 2 and 3 give the two distributions for $k=1000\ mc^2$, and show the region where they coincide, near $\kappa=0.1mc$. The total cross section obtained this way is $\sigma_{\text{Bors}}(k)-c_1$, where $c_1\equiv\int_0^{0.1}[\sigma_{\text{Bors}}(k,\kappa)-\sigma(k,\kappa)]d\kappa$, $\sigma_{\text{Bors}}(k,\kappa)$ is the Borsellino recoil distribution,[14] and $\sigma_{\text{Bors}}(k)$ is Borsellino's total triplet cross section.[15]

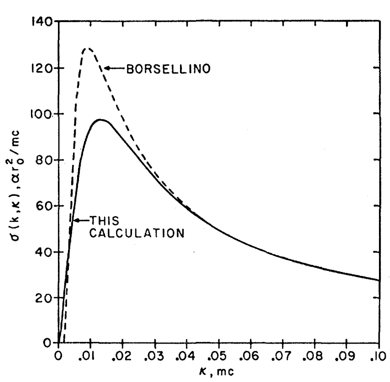

FIG. 2. Recoil distributions for photon energy 1000 mc^2. The difference between the curves in this region is due to neglect of binding in the Borsellino calculation.

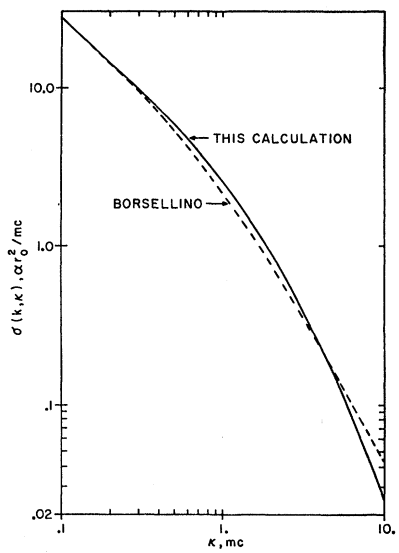

FIG. 3. Recoil distributions for photon energy 1000 mc^2. The difference between curves in this region is due to the neglect of retardation effects in the present calculation.

Table I gives the values of c_1 which were calculated. The estimated error is large because of the subtraction required to find c_1. Also shown in Table I is the estimate made by Gates[4,5] of the same correction, $c_2\equiv\sigma_{\text{BH}}-\sigma_{\text{WL}}$, where σ_{BH} is the total cross section for pair production in the field of an unscreened static proton and σ_{WL} is the total Wheeler-Lamb cross section for hydrogen. The measured total cross section[16] at 1 BeV is $(16.4\pm1.0)\alpha r_0^2$; $\sigma_{\text{Bors}}(1\text{ BeV})=17.4\alpha r_0^2$, so $\sigma_{\text{Bors}}-c_1=16.0\alpha r_0^2$, which is within experimental error.

ACKNOWLEDGMENTS

I want to thank Dr. Emil Kazes for his many ideas and suggestions concerning this problem. The cooperation of the NYU-AEC and Penn State computation centers is also gratefully acknowledged.

[13] We use the Borsellino distribution directly, rather than the Suh-Bethe approximation.
[14] Taken from Ref. 1, Eq. (42).
[15] Reference 1, Eq. (55).
[16] E. Malamud, Phys. Rev. **115**, 687 (1959).